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The Effect of Precompression on the Load-Deflection Relations of Long Rubber Bush Mountings

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Synopsis

For bonded cylindrical precompressed rubber bush mountings theoretical load-deflection relations are summarized for the four principal deformation modes of bushes which are sufficiently long so that end effects can be ignored. The relations are illustrated graphically for two types of initial compression.

INTRODUCTION

Rubber bush mountings consisting of cylindrical rubber tubes bonded on their outer and inner curved surfaces to effectively rigid metal cylinders are widely used as engineering components. Adkins and Gent[1] have considered four principal modes of deflection of initially unstressed rubber bushes, which they termed torsional, axial, radial, and tilting. The object of this work is to consider theoretically the effect of precompression on these modes of deflection for bushes which are sufficiently long so that end effects can be ignored.

The four principal modes of deflection are produced by fixing the outer metal cylinder while the inner one undergoes the following displacements: (i) a rotation about its axis (torsional), (ii) a translation in which each point moves parallel to the axis (axial), (iii) a translation in which each point moves through an equal distance in a radial direction (radial), and (iv) a rotation of the axis in a radial plane about a point on itself midway between the plane ends of the tube (tilting or conical).

In situations where large radial loads are expected, the rubber is precompressed on assembly in its housing. This effectively involves forcing the rubber tube over the inner cylinder and then forcing the outer cylinder over the rubber. The rubber is either "cold" bonded to the metal cylinders or, for situations where axial loads are not expected to be large, is not bonded at all. For no bonding, we assume that the initial compression is sufficiently large so as to prevent slipping between the rubber and the metal cylinders. We also assume that the initial compression can be approximated by a uniform radial inflation plus a simple extension parallel to the axis of the bush.

The problem of determining the load-deflection relations for the four principal modes of deflection superimposed upon this initial compression is mathematically a difficult one. However, for long bushes such relations can be derived which should at least provide some theoretical basis for the design of precompressed bushes. The torsional and axial relations are due to Rivlin,[3] while those for radial and tilting deformations have not been given previously and are derived in the Appendix. These relations are also applicable to bushes of finite lengths if suitable forces are applied to restrict the movement of the initially plane surfaces of the rubber tube.

In the following section, we define the geometry of the bush and the materials for which the subsequent results are applicable. We also give the relations describing the initial compression of the bush. In the section thereafter, we summarize existing formulae for the load–deflection relations of long bushes. We then show graphically the effect of compression on the four principal deformation modes for two different types of initial compression. Load–deflection relations for radial and tilting deformations are derived in the Appendix.

THEORETICAL PRELIMINARIES

We assume that the undeformed rubber tube has initially internal and external radii A and B, respectively, and is of length L. The material is assumed to be a homogeneous isotropic incompressible elastic material with strain-energy function $\sum$ given by the Mooney form

$$\sum = C_1(I_1 - 3) + C_2(I_2 - 3) \tag{1}$$

where I_1 and I_2 are the principal invariants of the finite deformation strain tensor and C_1 and C_2 are material constants. The linear shear modulus μ_0 is given by

$$\mu_0 = 2(C_1 + C_2). \tag{2}$$

We shall also consider the neo-Hookean material which has strain-energy function given by eq. (1) with $C_2 = 0$.

We suppose that the initial compression of the tube is effected by a deformation which increases the internal radius to a, decreases the external radius to b, and extends the length of the tube to l. These dimensions can be shown to be related to the initial dimensions of the tube by the equations

$$a^2 = \alpha A^2 + \beta, \qquad b^2 = \alpha B^2 + \beta, \qquad l = \frac{L}{\alpha} \tag{3}$$

where α and β are constants which we shall consider to be determined by the initial and final radii of the tube, that is,

$$\alpha = \frac{(b^2 - a^2)}{(B^2 - A^2)}, \qquad \beta = \frac{(a^2B^2 - b^2A^2)}{(B^2 - A^2)}. \tag{4}$$

Further details of this deformation are given in the Appendix.

LOAD–DEFLECTION RELATIONS

In this section, we give the load–deflection relations for long bushes for the four principal deformation modes which are superimposed on the initial compression.

(i) Torsional Deflections

Rivlin[3] has shown that the couple M required to rotate the inner metal cylinder about its axis through an ingle θ which is not necessarily small is given by

$$M = 4\pi\left(C_1 + \frac{C_2}{\alpha^2}\right)\frac{\beta L\theta}{\log\left(\dfrac{aB}{bA}\right)} \tag{5}$$

while for an initially unstressed rubber bush the couple M_0 is given by

$$M_0 = 4\pi\mu_0\frac{A^2B^2L\theta}{(B^2 - A^2)}. \tag{6}$$

We note that eq. (6) can be obtained from (5) be setting $\alpha = 1$ and letting β tend to zero.

(ii) Axial Deflections

Rivlin[3] has also shown that the force F required to displace the inner metal cylinder through a distance z parallel to its axis is given by

$$F = \frac{8\pi(C_1 + \alpha C_2)Lz}{\log\left(\dfrac{C_1B^2 + C_2b^2}{C_1A^2 + C_2a^2}\right)}. \tag{7}$$

The distance z is again not necessarily small, and from (7) we find that the force F_0 for an initially unstressed bush is given by

$$F_0 = \frac{2\pi\mu_0 Lz}{\log\left(\dfrac{B}{A}\right)}. \tag{8}$$

We remind the reader that eqs. (5) and (7) are strictly only valid for large strains for the case of long bushes and that, if applied to bushes of moderate lengths, departures from experimental findings will occur which will increase with increasing strains.

(iii) Radial Deflections

In the Appendix, we show that for small radial deformations the force G which is required to displace the inner metal cylinder a distance ϵ uniformly along its length in a radial direction is given by

$$G = 32\pi\left(C_1 + \frac{C_2}{\alpha^2}\right)\frac{K^2[(B^2+K)^2 - (A^2+K)^2]L\epsilon}{\{4(B^4 - A^4)[(B^2+K)^2 - (A^2+K)^2] - [\phi(B) - \phi(A)]^2\}} \tag{9}$$

where $K = \beta\alpha^{-1}$ and the function $\phi(R)$ is defined by

$$\phi(R) = K^2\log(R + \sqrt{R^2 + K}) - R\sqrt{R^2+K}\,(2R^2 + K). \tag{10}$$

For an initially unstressed bush, the force G_0 is given by[1]

$$G_0 = \frac{4\pi\mu_0(A^2 + B^2)L\epsilon}{[(A^2 + B^2)\log\dfrac{B}{A} - (B^2 - A^2)]} \tag{11}$$

and this result can also be derived from eq. (9) by taking the limit as K tends to zero.

(iv) Tilting or Conical Deflections

Also in the Appendix we show that for long bushes the couple N required to rotate the axis of the inner metal cylinder through a small angle δ about its mid-point is given by

$$N = \frac{1}{12}\left(\frac{G}{\epsilon}\right)\frac{L^2\delta}{\alpha^2} \tag{12}$$

where G/ϵ is obtained from eq. (9). Similarly for an unstressed bush, we have that the couple N_0 is given by

$$N_0 = \frac{1}{12}\left(\frac{G_0}{\epsilon}\right) L^2\delta \tag{13}$$

where G_0/ϵ is obtained from (11).

In the final section, we show graphically the variation with initial compression of the four ratios

$$m = \frac{M}{M_0}, \qquad f = \frac{F}{F_0}, \qquad g = \frac{G}{G_0}, \qquad n = \frac{N}{N_0}. \tag{14}$$

For the tilting ratio, we note that we have simply

$$n = \frac{g}{\alpha^2}. \tag{15}$$

NUMERICAL RESULTS

In order to illustrate the load–deflection relations of the previous section, we consider two types of initial compression. First, we suppose that the inner radius of the tube is increased while the outer is unchanged, so that $b = B$. If we define

$$x = \frac{A}{a}, \qquad \sigma = \frac{B}{A} \tag{16}$$

then from eq. (4) we obtain

$$\alpha = \frac{(\sigma^2x^2 - 1)}{x^2(\sigma^2 - 1)}, \qquad \beta = \frac{B^2(1 - x^2)}{x^2(\sigma^2 - 1)} \tag{17}$$

and since $a \geqslant A$ and $b > a$, we have

$$\frac{1}{\sigma} < x \leqslant 1. \tag{18}$$

For the second type of initial compression, we suppose that the inner radius is unchanged so that $a = A$ and the outer radius is decreased. In this case, we define

$$y = \frac{b}{B} \tag{19}$$

and again from (4) we obtain

$$\alpha = \frac{(\sigma^2y^2 - 1)}{(\sigma^2 - 1)}, \qquad \beta = \frac{B^2(1 - y^2)}{(\sigma^2 - 1)} \tag{20}$$

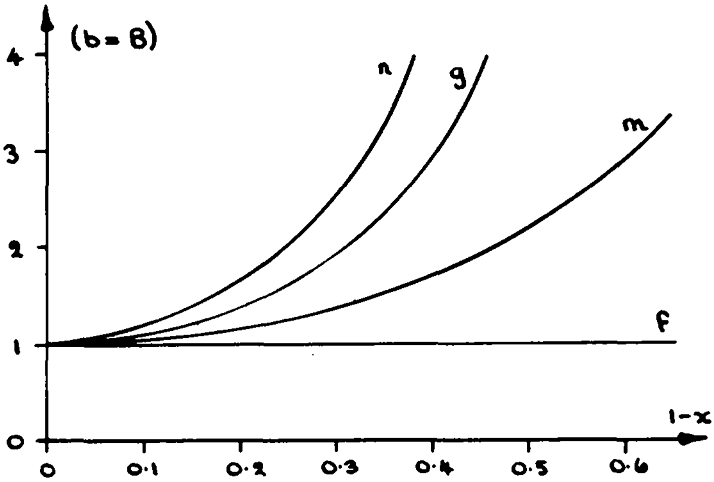

Fig. 1. Variation of the ratios m, f, g, and n with the first type of initial compression for a neo-Hookean material ($C_2 = 0$).

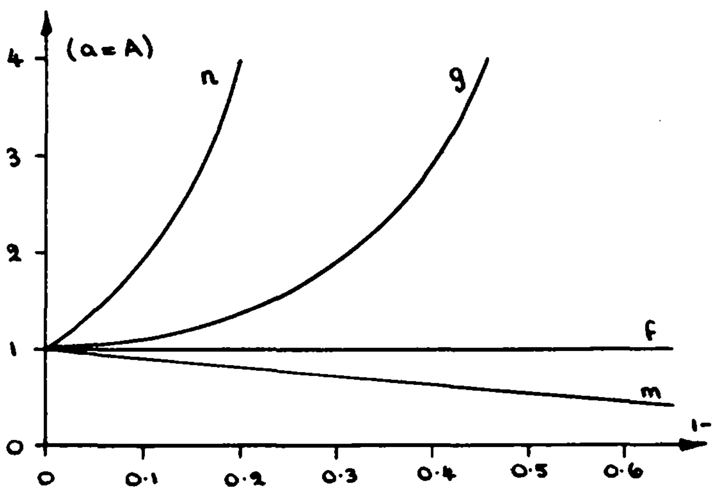

Fig. 2. Variation of the ratios m, f, g, and n with the second type of initial compression for a neo-Hookean material ($C_2 = 0$).

and we have

$$\frac{1}{\sigma} < y \leqslant 1. \tag{21}$$

For $A = 1$ and $B = 3$, Figures 1 and 2 show graphically the variation of the ratios m, g, and n with $1 - x$ and $1 - y$, respectively, for a neo-Hookean material. We note that for this material, $C_2 = 0$; and so from (7) and (8) we have $F = F_0$ and thus $f = 1$. For the same initial radii, Figures 3 and 4 give the variation of the ratios for a Mooney material with $C_2/C_1 = 0.1$, which is the value suggested by Rivlin and Thomas.[4]

In conclusion, for bonded cylindrical precompressed rubber bush mountings, we have summarized theoretical load–deflection relations for the four principal deformation modes. The torsional and axial relations are due to Rivlin,[3] while

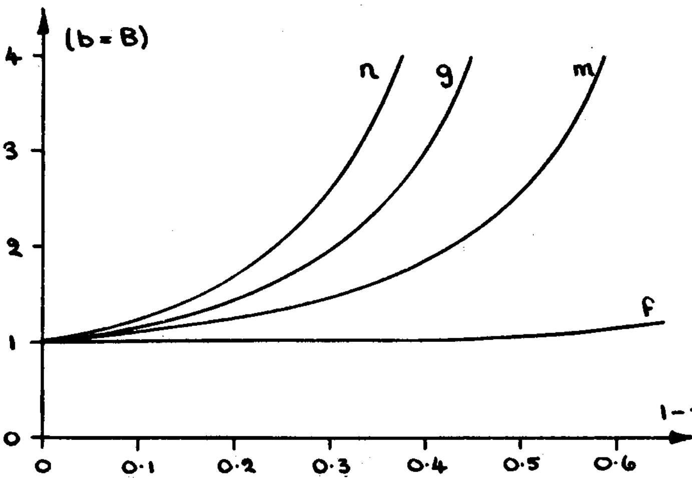

Fig. 3. Variation of the ratios m, f, g, and n with the first type of initial compression for a Mooney material ($C_2/C_1 = 0.1$).

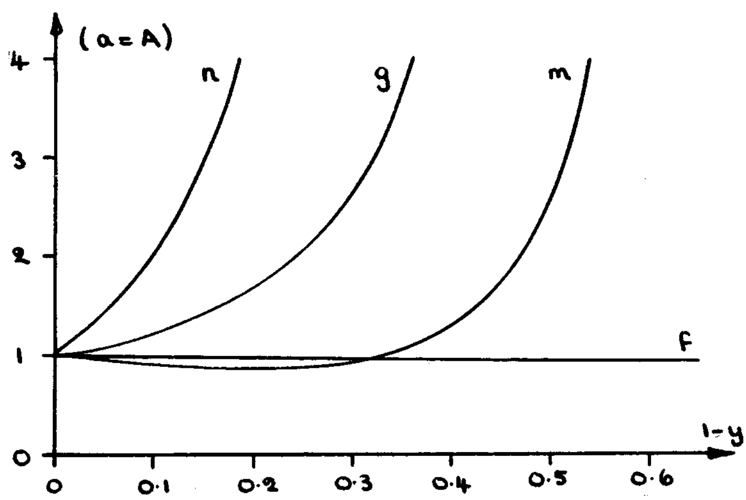

Fig. 4. Variation of the ratios m, f, g, and n with the second type of initial compression for a Mooney material ($C_2/C_1 = 0.1$).

those for radial and tilting deformations are derived in the Appendix. These relations apply to bushes which are sufficiently long so that end effects can be ignored, or to bushes of finite lengths if suitable forces are applied to the initially plane surfaces of the rubber tube to restrict the movement of these surfaces. The load–deflection relations are illustrated graphically for two types of initial compression for the neo-Hookean and Mooney materials.

Appendix

Before deriving the load–deflection relations for small radial and tilting deformations superimposed upon the initial compression, we summarize the general theory for large elastic plane deformations of a Mooney material. For material and spatial cylindrical polar coordinates (R, Θ, Z) and (r, θ, z), respectively, we consider the plane deformation

$$r = r(R, \Theta), \qquad \theta = \theta(R, \Theta), \qquad z = \frac{Z}{\alpha} \tag{A1}$$

where α is a positive constant. For an incompressible material, eqs. (A.1) satisfy the condition

$$r_R\theta_\Theta - r_\Theta\theta_R = \frac{\alpha R}{r} \tag{A2}$$

where r_R, θ_Θ, etc., denote partial derivatives. The equilibrium equations for an isotropic incompressible Mooney material can be shown to become[3]

$$p_r^* = \mu\left\{\nabla^2 r - r\left(\theta_R^2 + \frac{\theta_\Theta^2}{R^2}\right)\right\}$$

$$p_\theta^* = \mu r^2\left\{\nabla^2\theta + \frac{2}{r}\left(r_R\theta_R + \frac{r_\Theta\theta_\Theta}{R^2}\right)\right\} \tag{A3}$$

where p^* is the pressure function associated with incompressible materials, μ is a constant given by

$$\mu = 2\left(C_1 + \frac{C_2}{\alpha^2}\right) \tag{A4}$$

where C_1 and C_2 are the Mooney constants, and ∇^2 is the two-dimensional Laplacian, that is,

$$\nabla^2 = \frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial^2}{\partial\Theta^2}. \tag{A5}$$

The contravariant components of the Cauchy stress tensor are given by

$$\begin{aligned} t^{11} &= -p^* + \mu\left(r_R^2 + \frac{r_\Theta^2}{R^2}\right) \\ t^{22} &= -\frac{p^*}{r^2} + \mu\left(\theta_R^2 + \frac{\theta_\Theta^2}{R^2}\right) \\ t^{12} &= \mu\left(r_R\theta_R + \frac{r_\Theta\theta_\Theta}{R^2}\right) \\ t^{33} &= -p^* + \frac{2}{\alpha^2}[C_1 + C_2(\mathrm{I} - \alpha^4)] \end{aligned} \tag{A6}$$

where I is given by

$$I = r_R^2 + \frac{r_\Theta^2}{R^2} + r^2\left(\theta_R^2 + \frac{\theta_\Theta^2}{R^2}\right) \tag{A7}$$

and all other components of the stress tensor are zero. Using (A6), we can deduce that the force G which must be applied in the direction $\Theta = 0$ to a cylinder which is given originally by the circular cylinder R = constant is given by

$$G = -\frac{L}{\alpha}\int_0^{2\pi}[-p^*(r\sin\theta)_\Theta + \mu\alpha R(r\cos\theta)_R]d\Theta \tag{A8}$$

where L is the original length of the cylinder.

The initial compression of the rubber tube as described in the section on theoretical preliminaries is assumed to be effected by the deformation

$$r = \sqrt{\alpha R^2 + \beta}, \qquad \theta = \Theta, \qquad z = \frac{Z}{\alpha} \tag{A9}$$

where α and β are constants given by (4). From (A3), we find that the pressure function for this deformation is given by

$$p_0(R) = \frac{\mu\alpha}{2}\left\{\log\frac{(\alpha R^2 + \beta)}{R^2} - \frac{\beta}{(\alpha R^2 + \beta)}\right\} + \rho \tag{A10}$$

where ρ is a constant. The stress tensor $t_0{}^{ij}$ corresponding to (A9) is now easily obtained from (A6), and we assume that the constant ρ is determined by the condition

$$\int_a^b t_0{}^{33}\, r\, dr = 0 \tag{A11}$$

which approximates the boundary condition of no surface tractions on the plane ends of the bush.

For small radial deformations superimposed upon this initial compression, the displacement boundary conditions on the inner cylinder suggest we look for solutions of (A2) and (A3) of the form

$$\begin{aligned} r &= \sqrt{\alpha R^2 + \beta} + \epsilon u(R) \cos \Theta \\ \theta &= \Theta + \epsilon \frac{v(R)}{\sqrt{\alpha}} \sin \Theta \\ p^* &= p_0(R) + \epsilon \sqrt{\alpha} p(R) \cos \Theta \end{aligned} \tag{A12}$$

where u, v, and p are functions of R only and ϵ is the distance moved by the inner cylinder in the direction $\Theta = 0$. The displacement boundary conditions at the inner and outer cylinders now become

$$\begin{aligned} u(A) &= 1, \qquad v(A) = -\frac{1}{\sqrt{A^2 + K}} \\ u(B) &= 0, \qquad v(B) = 0 \end{aligned} \tag{A13}$$

where $K = \beta\alpha^{-1}$. In the physical stituation which is of interest, we have $a \geqslant A$ and $b \leqslant B$, from which we can deduce using eq. (4) that $K \geqslant 0$, and we shall assume throughout that this is the case. From (A2) and (A3) we obtain, on equating terms of order ϵ, the following ordinary differential equations for the functions $u(R)$, $v(R)$, and $p(R)$:

$$\begin{aligned} &\frac{\sqrt{R^2 + K}}{R} u' + \frac{u}{\sqrt{R^2 + K}} + v = 0 \\ p' &= \mu \left\{ \frac{R}{\sqrt{R^2 + K}} \left[u'' + \frac{u'}{R} - \frac{2u}{R^2} - \frac{2}{R^2} \sqrt{R^2 + K} v \right] - \frac{K^2 u'}{R^3 (R^2 + K)^{3/2}} \right\} \\ -p &= \mu \left\{ (R^2 + K) \left[v'' + \frac{v'}{R} - \frac{v}{R^2} + \frac{2Rv'}{(R^2 + K)} - \frac{2u}{R^2 \sqrt{R^2 + K}} \right] + \frac{K^2 u}{R^2 (R^2 + K)^{3/2}} \right\} \end{aligned} \tag{A14}$$

where primes denote differentiation with respect to R. It can be verified that these equations have the following solutions:

$$\begin{aligned} u(R) &= \gamma_1 \left\{ R^2 - \frac{KR}{\sqrt{R^2 + K}} \log \left(R + \sqrt{R^2 + K}\right) \right\} \\ &\quad + \gamma_2 \{ R\sqrt{R^2 + K} + K \log (R + \sqrt{R^2 + K}) \} + \frac{\gamma_3 R}{\sqrt{R^2 + K}} + \gamma_4 \\ v(R) &= \gamma_1 \left\{ \frac{K}{R} \log (R + \sqrt{R^2 + K}) - \frac{(3R^2 + K)}{\sqrt{R^2 + K}} \right\} \\ &\quad - \gamma_2 \left\{ 3R + \frac{2K}{R} + \frac{K}{\sqrt{R^2 + K}} \log (R + \sqrt{R^2 + K}) \right\} - \frac{\gamma_3}{R} - \frac{\gamma_4}{\sqrt{R^2 + K}} \\ p(R) &= \gamma_1 \mu \left\{ \frac{K^3}{R(R^2 + K)^2} \log (R + \sqrt{R^2 + K}) + \frac{R^4}{(R^2 + K)^{3/2}} + 7\sqrt{R^2 + K} \right\} \\ &\quad + 2\gamma_2 \frac{\mu}{R} (4R^2 - K) - \frac{\gamma_3 \mu K^2}{R(R^2 + K)^2} \end{aligned} \tag{A15}$$

where γ_1, γ_2, γ_3, and γ_4 are four arbitrary constants which are determined by the boundary conditions (A13).

Now, from (A8) and (A12) we find that the force G required to maintain the deformation is given by

$$G = -\epsilon\pi L \{-p(R)\sqrt{R^2 + K} + \mu R\, [u(R) - v(R)\sqrt{R^2 + K}]'\} \tag{A16}$$

which on using the solutions (A15) can be shown to become

$$G = 8\pi\mu\epsilon K L \gamma_1. \tag{A17}$$

From the solutions (A15) and the boundary conditions (A13), we find that γ_1 is given by

$$\gamma_1 = \frac{2K[(B^2 + K)^2 - (A^2 + K)^2]}{\{4(B^4 - A^4)[(B^2 + K)^2 - (A^2 + K)^2] - [\phi(B) - \phi(A)]^2\}} \tag{A18}$$

where the function $\phi(R)$ is defined by (10). Thus from (A4), (A17) and (A18) we have the load-deflection relation (9) for radial deformations.

In order to obtain the load–deflection relation for small superimposed tilting deflections of long rubber bushes, we use the argument employed by Adkins and Gent[1] for unstressed bushes. If the inner metal cylinder of the bush is rotated through a small angle δ about its midpoint, then formally to obtain the couple N which is required to maintain the deformation we replace ϵ by δz and L by dZ in (A17) and integrate the moment $z\, dG$ over the length of the bush, that is,

$$N = 8\pi\mu K\delta\gamma_1 \int_{-L/2}^{L/2} z^2\, dZ \tag{A19}$$

which, on using $(A1)_3$ and (A17), yields the load–deflection relation (12). The reader is referred to Adkins and Gent[1] for the precise details of the argument.

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